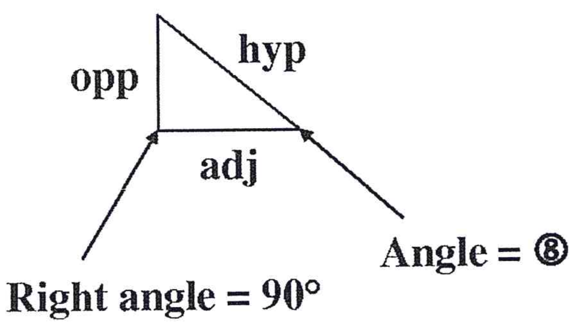
13.1 Right Triangle Trigonometry



sohcahtoa

$$\text{Sin}\,\theta = \frac{\text{opp}}{\text{hyp}} \quad \text{Cos}\,\theta = \frac{\text{adj}}{\text{hyp}} \quad \text{Tan}\,\theta = \frac{\text{opp}}{\text{adj}}$$

$$\text{Csc}\,\theta = \frac{\text{hyp}}{\text{opp}} \quad \text{Sec}\,\theta = \frac{\text{hyp}}{\text{adj}} \quad \text{Cot}\,\theta = \frac{\text{adj}}{\text{opp}}$$

- Adjacent **is the side that is touching angle ⑧.**
- Opposite **is the side that does not touch ⑧.**
- Hypotenuse **is the diagonal touching ⑧.**

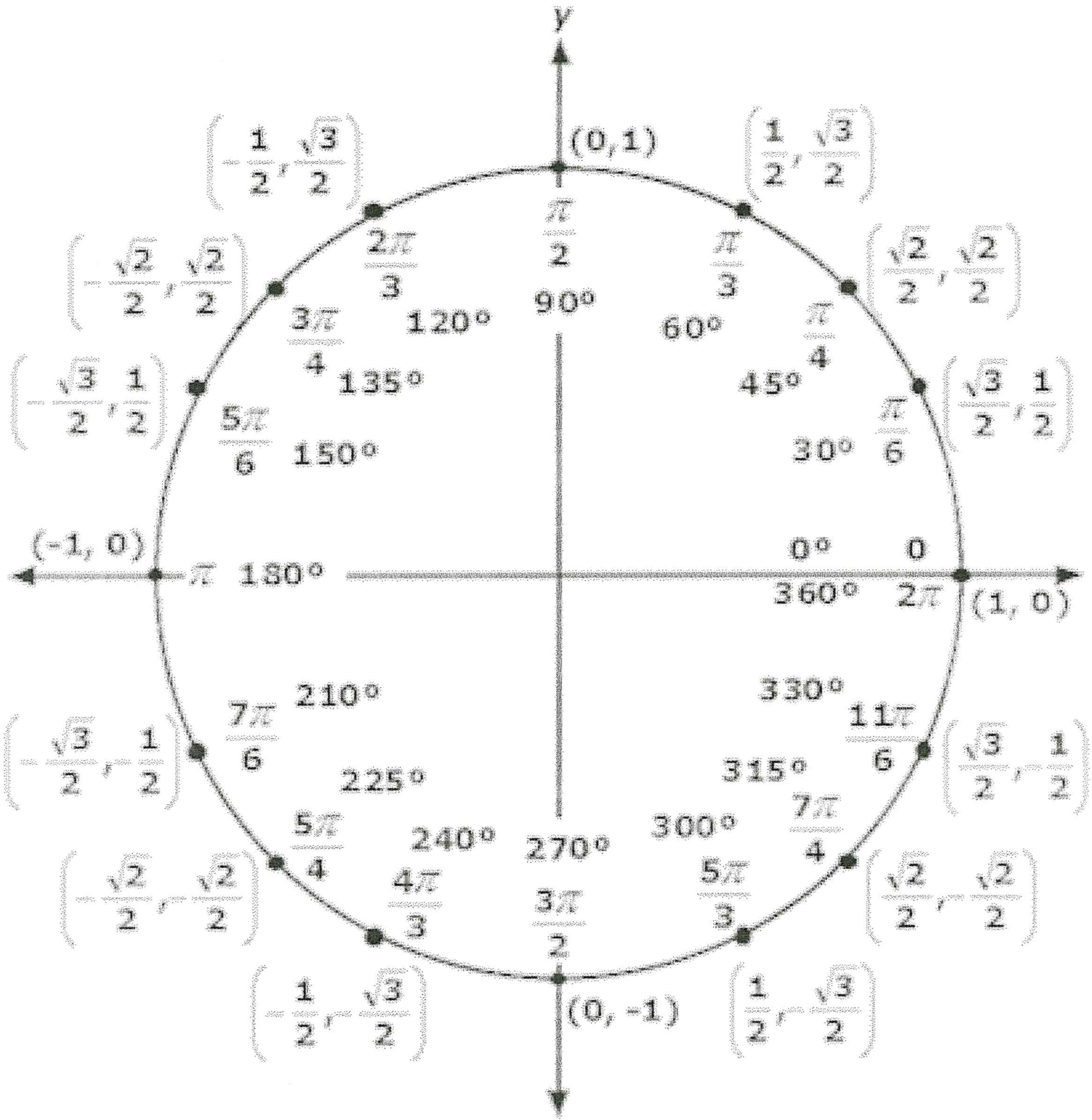
Pythagorean Theorem:

$$\text{adj}^2 + \text{opp}^2 = \text{hyp}^2$$

$$a^2 + b^2 = c^2$$

All six ratios are defined as follows. Let θ be an acute angle of a right triangle. Refer to Fig. 1.2. Then:

Name	Notation	Definition
sine	sin	$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
cosine	cos	$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
tangent	tan	$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{\sin\theta}{\cos\theta}$
cotangent	cot	$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\text{adjacent}}{\text{opposite}}$
secant	sec	$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
cosecant	csc	$\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{\text{hypotenuse}}{\text{opposite}}$



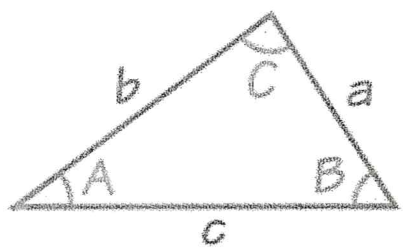
y
(0,1)
(1, 0)
(-1, 0)
(0, -1)
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\frac{\pi}{2}$ 90°
$\frac{\pi}{3}$ 60°
$\frac{\pi}{4}$ 45°
$\frac{\pi}{6}$ 30°
0° 0
360° 2π
$\frac{2\pi}{3}$ 120°
$\frac{3\pi}{4}$ 135°
$\frac{5\pi}{6}$ 150°
π 180°
210° $\frac{7\pi}{6}$
225° $\frac{5\pi}{4}$
240° $\frac{4\pi}{3}$
270° $\frac{3\pi}{2}$
300° $\frac{5\pi}{3}$
315° $\frac{7\pi}{4}$
330° $\frac{11\pi}{6}$

The Law of Sines

The Law of Sines (or **Sine Rule**) is very useful for solving triangles:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

It works for any triangle:



a, **b** and **c** are sides.

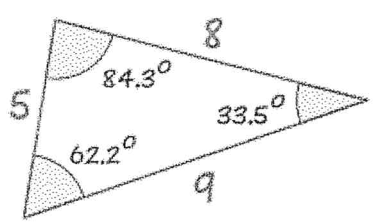
A, **B** and **C** are angles.

(Side a faces angle A, side b faces angle B and side c faces angle C).

So if you **divide side a by the sine of angle A** it is equal to **side b divided by the sine of angle B**, and also equal to **side c divided by the sine of angle C**

Sure ... ?

Well, let's do the calculations for a triangle I prepared earlier:



a/sin A = 8 / sin (62.2°) = 8 / 0.885... = **9.04...**

b/sin B = 5 / sin (33.5°) = 5 / 0.552... = **9.06...**

c/sin C = 9 / sin (84.3°) = 9 / 0.995... = **9.05...**

The answers are **almost the same!**
*(They would be **exactly** the same if I used perfect accuracy).*

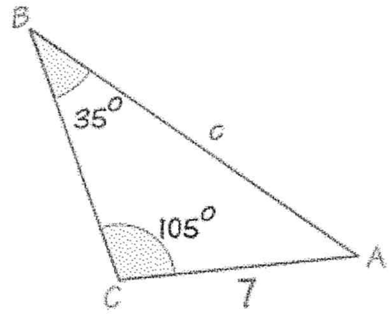
So now you can see that:

a/sin A = b/sin B = c/sin C

How Do I Use It?

Let us see an example:

Example: Calculate side "c"



Law of Sines: a/sin A = b/sin B = c/sin C

Put in the values we know: a/sin A = 7/sin(35°) = c/sin(105°)

Ignore a/sin A (not useful to us): 7/sin(35°) = c/sin(105°)

Now we use our algebra skills to rearrange and solve:

Swap sides: c/sin(105°) = 7/sin(35°)

Multiply both sides by sin(105°): c = (7 / sin(35°)) × sin(105°)

Calculate: c = (7 / 0.574...) × 0.966...

Calculate: c = **11.8** (to 1 decimal place)

Finding an Unknown Angle

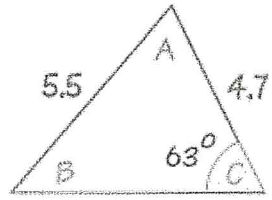
In the previous example we found an unknown side ...

... but we can also use the Law of Sines to find an **unknown angle**.

In this case it is best to turn the fractions upside down (**sin A/a** instead of **a/sin A**, etc):

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example: Calculate angle B



Start with: sin A / a = sin B / b = sin C / c

Put in the values we know: sin A / a = sin B / 4.7 = sin(63°) / 5.5

Ignore "sin A / a": sin B / 4.7 = sin(63°) / 5.5

Multiply both sides by 4.7: sin B = (sin63°/5.5) × 4.7

Calculate: sin B = 0.7614...

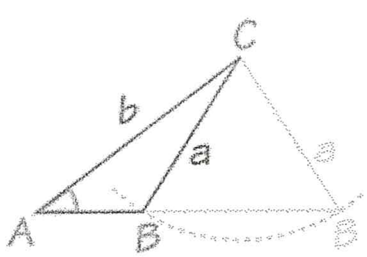
Inverse Sine: B = $\sin^{-1}$(0.7614...)

B = **49.6°**

Sometimes There Are Two Answers !

There is one **very** tricky thing you have to look out for:

Two possible answers.



Let us say you know angle **A**, and sides **a** and **b**.

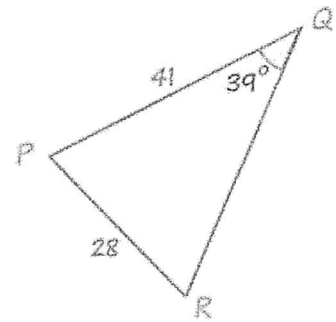
You could swing side **a** to left or right and come up with two possible results (a small triangle and a much wider triangle)

Both answers are right!

This only happens in the "Two Sides and an Angle **not** between" case, and even then not always, but you have to watch out for it.

Just think "could I swing that side the other way to also make a correct answer?"

Example: Calculate angle R



The first thing to notice is that this triangle has different labels: PQR instead of ABC. But that's OK. We just use P,Q and R instead of A, B and C in The Law of Sines.

Start with: sin R / r = sin Q / q

Put in the values we know: sin R / 41 = sin(39°)/28

Multiply both sides by 41: sin R = (sin39°/28) × 41

Calculate: sin R = 0.9215...

Inverse Sine: R = $\sin^{-1}$(0.9215...)

R = **67.1°**

But wait! There's another angle that also has a sine equal to 0.9215...

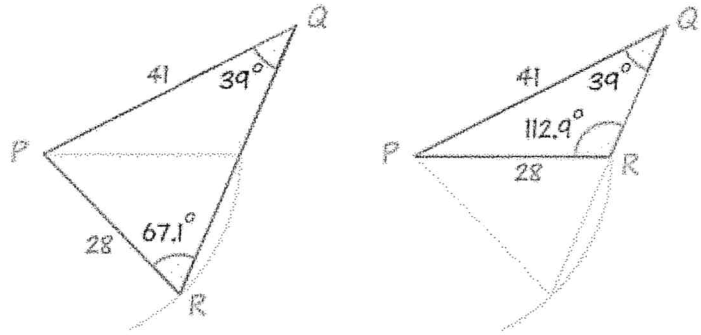
Your calculator won't tell you this but sin(112.9°) is also equal to 0.9215... (try it!)

So ... how do you discover the vale 112.9°?

Easy ... take 67.1° away from 180°, like this:

$$180° - 67.1° = 112.9°$$

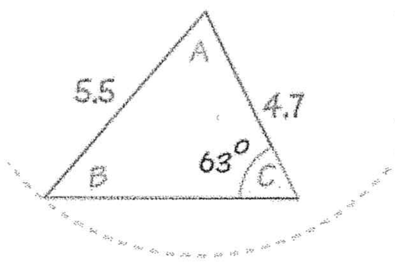
So there are two possible answers for R: **67.1°** and **112.9°**:



Both are possible! Each one has the 39° angle, and sides of 41 and 28.

So, always check to see whether the alternative answer makes sense.

- ... sometimes it will (like above) and there will be **two solutions**
- ... sometimes it won't (see below) and there is **one solution**



We looked at this triangle before.

As you can see, you can try swinging the "5.5" line around, but no other solution makes sense.

So this has only one solution.

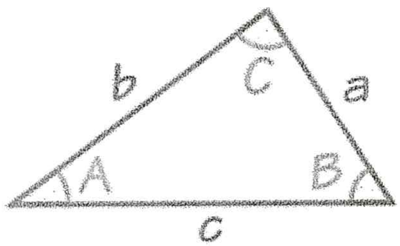
Question 1 Question 2 Question 3 Question 4 Question 5
Question 6 Question 7 Question 8 Question 9 Question 10

The Law of Cosines

The Law of Cosines (also called the **Cosine Rule**) is very useful for solving triangles:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

It works for any triangle:

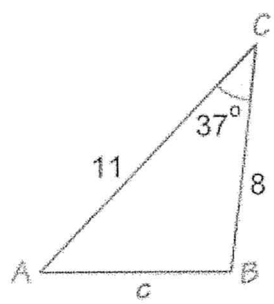


a, **b** and **c** are sides.

C is the angle opposite side c

Let's see how to use it in an example:

Example: How long is side "c" ... ?



We know angle C = 37°, a = 8 and b = 11

The Law of Cosines says: $c^2 = a^2 + b^2 - 2ab \cos(C)$

Put in the values we know: $c^2 = 8^2 + 11^2 - 2 \times 8 \times 11 \times \cos(37^\circ)$

Do some calculations: $c^2 = 64 + 121 - 176 \times 0.798...$

Which gives us: $c^2 = 44.44...$

Take the square root: $c = \sqrt{44.44} = 6.67$ (to 2 decimal places)

Answer: c = 6.67

How to Remember

How can you remember the formula?

Well, it helps to know it's the Pythagoras Theorem with something extra so it works for all triangles:

Pythagoras Theorem: $a^2 + b^2 = c^2$ (only for Right-Angled Triangles)

Law of Cosines: $a^2 + b^2 - 2ab\cos(C) = c^2$ (for all triangles)

So, to remember it:

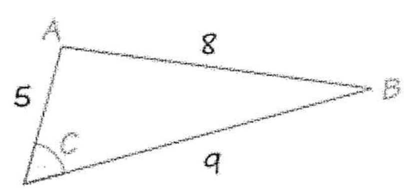
- think "**abc**": $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2$,
- then a **2nd** "**abc**": $2ab\cos(C)$,
- and put them together: $\mathbf{a^2 + b^2 - 2ab\cos(C) = c^2}$

When to Use

The Law of Cosines is useful for finding:

- the third side of a triangle when you know **two sides and the angle between** them (like the example above)
- the angles of a triangle when you know **all three sides** (as in the following example)

Example: What is Angle "C" ...?



The side of length "8" is opposite angle ***C***, so it is side **c**. The other two sides are **a** and **b**.

Now let us put what we know into **The Law of Cosines**:

Start with: $c^2 = a^2 + b^2 - 2ab\cos(C)$

Put in a, b and c $8^2 = 9^2 + 5^2 - 2 \times 9 \times 5 \times \cos(C)$

Calculate: $64 = 81 + 25 - 90 \times \cos(C)$

Calculate some more: 64 = 106 - 90 × cos(C)

Now we use our algebra skills to rearrange and solve:

Subtract 64 from both sides: 0 = 42 - 90 × cos(C)

Add "90 × cos(C)" to both sides: 90 × cos(C) = 42

Divide both sides by 90: cos(C) = 42/90

Inverse cosine: C = $\cos^{-1}(42/90)$

Calculator: C = **62.2°** (to 1 decimal place)

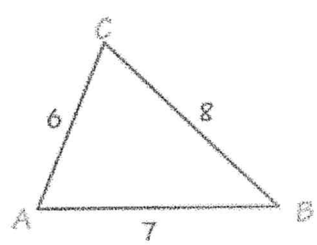
In Other Forms

Easier Version For Angles

We just saw how to find an angle when we know three sides. It took quite a few steps, so it may help you to know the "direct" formula (which is just a rearrangement of the $c^2 = a^2 + b^2 - 2ab\cos(C)$ formula):

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example: Find Angle "C" Using The Law of Cosines (angle version)



In this triangle we know the three sides:

- a = 8,
- b = 6 and
- c = 7.

Use The Law of Cosines (angle version) to find angle **C** :

$\cos C = (a^2 + b^2 - c^2)/2ab$

$= (8^2 + 6^2 - 7^2)/2\times8\times6 = (64 + 36 - 49)/96 = 51/96 = 0.53125$

$C = \cos^{-1}(0.53125)$

= **57.9°** correct to one decimal place.

Versions for a, b and c

Also, you can rewrite the $c^2 = a^2 + b^2 - 2ab\cos(C)$ formula into "a^2=" and "b^2=" form.

Here are all three:

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

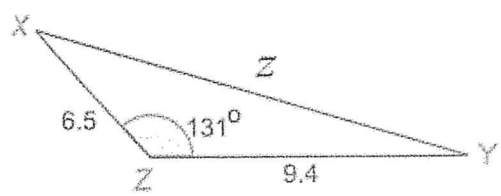
$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

But it is easier to remember the "$\mathbf{c^2}$=" form and change the letters as needed !

As in this example:

Example: Find the distance "z"



The letters are different! But that doesn't matter. We can easily substitute x for a, y for b and z for c

Start with: $c^2 = a^2 + b^2 - 2ab\cos(C)$

x for a, y for b and z for c $z^2 = x^2 + y^2 - 2xy\cos(Z)$

Put in the values we know: $z^2 = 9.4^2 + 6.5^2 - 2\times9.4\times6.5\times\cos(131°)$

Calculate: $z^2 = 88.36 + 42.25 - 122.2\times(-0.656...)$

$z^2 = 130.61 + 80.17...$

$z^2 = 210.78...$

$z = \sqrt{210.78...} = 14.5$ to 1 decimal place.

Answer: $z = 14.5$

Did you notice that cos(131°) is negative and this changes the last sign in the calculation to + (plus)? The cosine of an obtuse angle is always negative (see Unit Circle).