Rules of Exponents

- Product Rule: $b^m \cdot b^n = b^{m+n}$
- Quotient Rule: $\frac{b^m}{b^n} = b^{m-n}$
- Negative Exponent: $b^{-m} = \frac{1}{b^m}$
- Zero Exponent: $b^0 = 1, \quad b \neq 0$

At this point, it is useful to discuss the rules of exponents in preparation for solving exponential equations.

First, the Product Rule tells us that when multiplying exponential expressions with the same base, we add the exponents. Similarly, when taking the quotient of two exponential expressions with the same base, we take the difference of the exponents of the terms as the exponent of the resulting expression. The negative exponent rule shows how to evaluate expressions with negative exponents. This rule can be extended to the case where a term with a negative exponent appears in the denominator: 1/b^-m = 1/(/1/b^m) =b^m.

The zero exponent rule indicates that for any base other than 0, b^0 =1.

Rules of Exponents

- Power Rule: $\left(b^m\right)^n = b^{mn}$
- Power of a Product: $\left(ab\right)^m = a^m b^m$
- Power of a Quotient: $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

The next three rules are applicable when raising an exponential expression to another power. If an exponential term is raised to another power, the Power Rule states to take the product of the exponents.

The final two rules, the Power of a Product and the Power of a Quotient rules indicate that an exponent is applied to each term in a product or a quotient. This same principle does NOT apply to sums and differences, however.

Rational exponents:

$$b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^{m}$$

Rational exponents represent a special case in the discussion of exponential expressions. Rational exponent notation is actually an alternate way of writing expressions involving radicals. The denominator of the rational exponent represents the index of the root that is to be taken, and the numerator represents the actual power that this root is to be raised to.

For example, 16^3/4 is equivalent to the fourth root of 16, raised to the third power. The fourth root of 16 is 2. Completing the simplification, we end up with 8.

Exponential Parent Functions

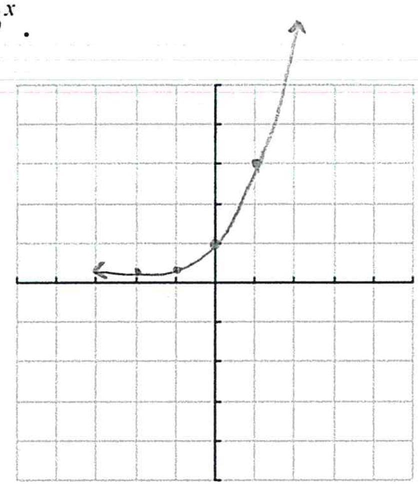
- *Exponential parent functions* are functions of the form
$$f(x) = b^x,\ b > 1 \text{ or } 0 < b < 1$$
- The base b determines the direction of the graph.

Two additional functions will be added to the list of parent functions, one of these being the exponential function.

Exponential parent functions are functions of the form f(x) = b^x, where b is known as the base of the function. The base must fall into one of two ranges of values: either b is greater than 1 or b is between 0 and 1. 1 is excluded from the set of values for b, since b=1 would correspond to a constant function. The value of the base of an exponential parental function determines the direction of the graph.

If $b > 1$, $f(x) = b^x$ increases from left to right.

Example: Graph $f(x) = 3^x$.



If the base is greater than 1, then the graph of the function increases from left to right. Notice the general shape of the graph. The graph begins increase more and more rapidly as x increases.

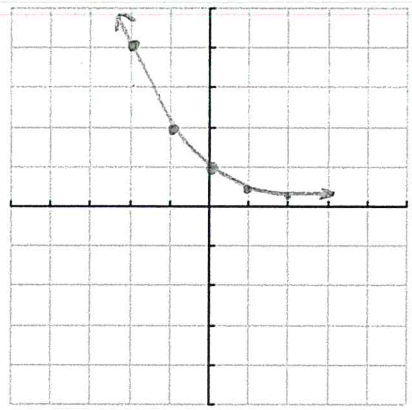
It is easy to see that the domain is defined for all real numbers. However, the range of this function is composed of all positive numbers.

$f(x) = 3^x$

x	y	
-2	$\frac{1}{9}$	.11
-1	1/3	.33
0	1	
1	3	
2	9	

If $0 < b < 1$, $f(x) = b^x$ decreases from left to right.

Example: Graph $f(x) = \left(\frac{1}{2}\right)^x$.



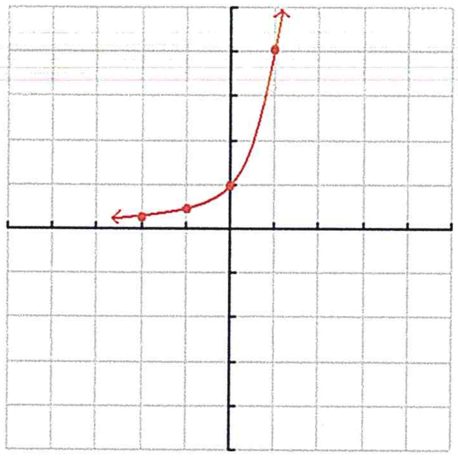
The graph of an exponential function decreases if the base falls between 0 and 1.

The graph of this function looks almost like a mirror image of the graph of the previous function.

The domain and range are the same for this function.

x	y	
-2	4	
-1	2	
0	1	
1	1/2	.5
2	1/4	.25

Example: Identify the base of the following exponential function.



The graph of an exponential parent function can be used to identify the base of the function. Remember that at x = 1, b^1 = b. So the base of this function can be found by locating the point corresponding to x =1. This point occurs at (1,4). So b^1 = 4. Thus the base of this exponential function is 4.

find $x=1$

at $x=1$, $y=4$

$f(x)=4^x$

Transformations of Exponential Functions

- Horizontal shifts
  - $g(x) = b^{x+d}$
  - $g(x) = b^{x-d}$

left and right

- Vertical shifts
  - $g(x) = b^{x} + c$
  - $g(x) = b^{x} - c$

up and down

Additional, more complex exponential functions can be graphed by applying transformations previously seen to the exponential parent functions.

Horizontal shifts behave in the same way as before. Replacing x in the exponent with x+d shifts the graph of f(x) = b^x to the left d units. Replacing x with x-d shifts the graph of the parent function to the right h units.

Vertical shifts are easily identified. Adding a value c to f(x) = b^x shifts the graph upward c units, and subtracting a value c from the parent function shifts the graph downward c units.

Transformations of Exponential Functions

- Reflections
  - $g(x) = b^{-x}$
  - $g(x) = -b^{x}$
- Vertical stretch or compression
  - $g(x) = \mathbf{a} \cdot b^{x}$

Exponential parent functions may also be reflected about the y-axis by replacing x with –x. If b^x is multiplied by -1, then the graph of the parent function is reflected about the x-axis.

An exponential parent function may also be stretched or compressed vertically by a factor of a.

$a > 1$ stretch vertically

$0 < a < 1$ compress vertically

$a < 0$ reflect in the x-axis

Horizontal Stretch of an Exponential Function

Horizontal Stretch

$y = 2^{\frac{x}{c}}$

- Another transformation that can be made to the base function $y = 2^x$ is to change the **Horizontal Stretch** (HS). The horizontal stretch affects the graph by stretching (or compressing) the graph horizontally. The result will produce an image graph that looks steeper or less steep than the original base graph.
- Example 1:
  Let's look at the following functions. We will use a mapping rule to generate a table of values for the image function and then graph all three functions on the same axes.

(a.) $y = 2^x$

(b.) $y = 2^{2x}$

$(x,\ y) \rightarrow (\frac{1}{2}x,\ y)$

(c.) $y = 2^{\frac{x}{2}}$

$(x,\ y) \rightarrow (2x,\ y)$

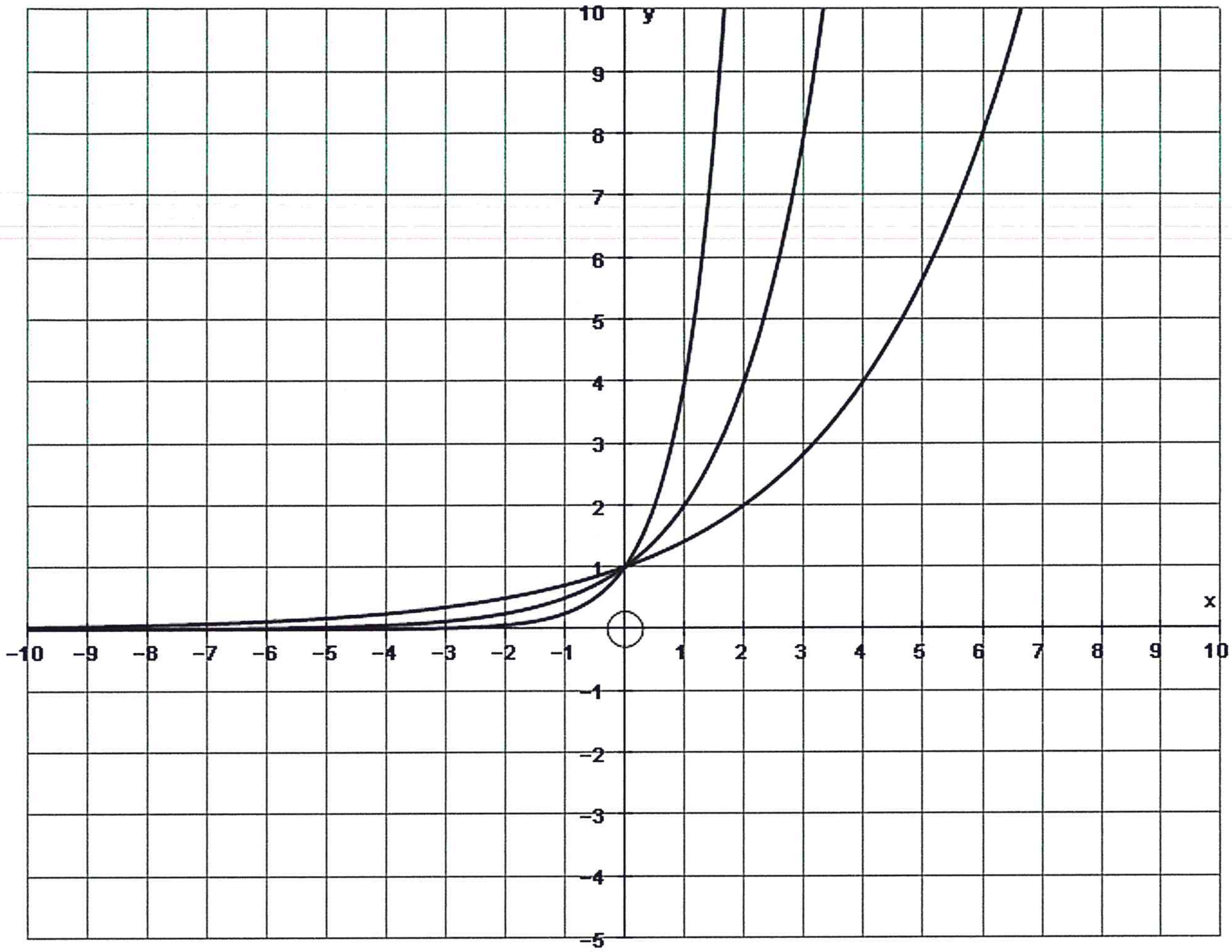
x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

x	y
$-\frac{3}{2}$	$\frac{1}{8}$
-1	$\frac{1}{4}$
$-\frac{1}{2}$	$\frac{1}{2}$
0	1
$\frac{1}{2}$	2
1	4
$\frac{3}{2}$	8

x	y
-6	$\frac{1}{8}$
-4	$\frac{1}{4}$
-2	$\frac{1}{2}$
0	1
2	2
4	4
6	8

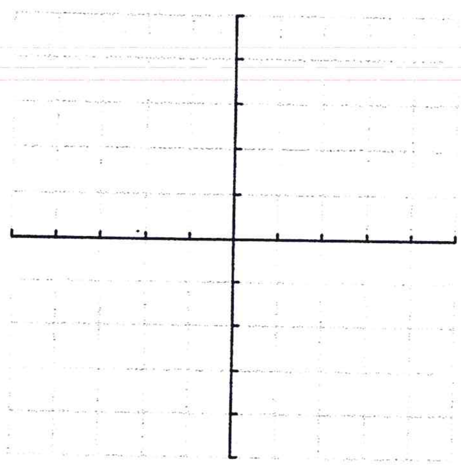
$K > 1$ horizontal compression

$0 < K < 1$ horizontal stretch



- When the equation is $y = 2^{2x}$, then there is a horizontal stretch of $\frac{1}{2}$. This will result in a graph that is **"steeper"** than the graph of $y = 2^x$.
- When the equation is $y = 2^{\frac{x}{2}}$, then there is a horizontal stretch of 2. This will result in a graph that is **"less steep"** than the graph of $y = 2^x$.
- The horizontal stretch factor is the **reciprocal** of the coefficient of x in the function.

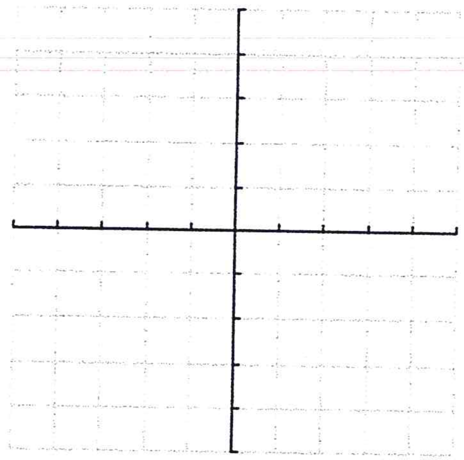
Example: Graph $g(x)=2\left(3^{x+1}\right)$.



Example:

Following the previously graphed transformations of functions, start by identifying the parent function. The parent function in this case is f(x) =3^x. Two transformations have then been applied: first the graph of f(x) = 3^x has been shifted left 1 unit and then stretched vertically by a factor of 2.

Example: Graph $F(x) = -\left(\frac{1}{2}\right)^x - 2$.



Example:

The parent function here is another familiar exponential function: f(x) = (1/2)^x. Once again, we have two transformations: reflection about the x-axis and shifted down 2 units.

Notice that the range has changed after the transformations have been applied, but the domain remains the same.