

Exponential Growth

3.1

- A bacteria grows so that it triples in number every day. On the day Roger begins observing the bacteria, a sample has a population of 100.

- 1) Find the population after each of the 1st 4 days.

Day	Population
0	100
1	$100 \times 3 = 300$
2	$300 \times 3 = 900$
3	$900 \times 3 = 2700$
4	$2700 \times 3 = 8100$

- 2) Write an equation to model this growth.

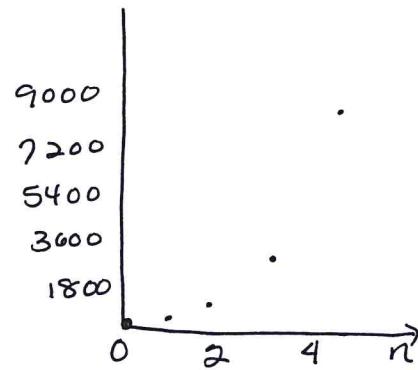
- To do this express each population calculation in terms of the number of times the initial population is tripled.

Day	Population
0	100
1	$100 \times 3^1 = 300$
2	$100 \times 3^2 = 900$
3	$100 \times 3^3 = 2700$
4	$100 \times 3^4 = 8100$
n	100×3^n

$$P(n) = 100 \times 3^n$$

3) Graph $P(n) = 100 \times 3^n$

is it a function?



Yes → each element

in the domain ^(x) corresponds
to exactly one element in
the range ^(y)

4) Predict the population after 2 weeks.

$$P(n) = 100 \times 3^n \quad n = 14 \text{ days}$$

$$\begin{aligned} P(14) &= 100 \times 3^{14} \\ &= 478,296,900 \end{aligned}$$

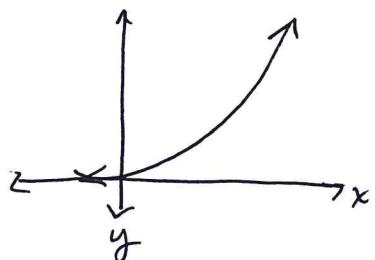
5) Describe the pattern of finite differences

DAY	Population	1st Dif	2nd Dif
0	100	$300 - 100 = 200$	$600 - 200 = 400$
1	300	$900 - 300 = 600$	$1800 - 600 = 1200$
2	900	$2700 - 900 = 1800$	$5400 - 1800 = 3600$
3	2700	$8100 - 2700 = 5400$	
4	8100		

not constant but a constant ratio : 3 times the previous value.

Exponential Growth

- pattern of growth in which each term is multiplied by a constant amount (greater than 1) to produce the next term
- produces a graph that increases at a constantly increasing rate.
- the ratio of successive finite differences is constant



- Any number raised to the exponent zero = 1

Laws of Exponents

A power involving a negative exponent can be expressed using a positive exponent

$$b^{-n} = \frac{1}{b^n} \quad b \neq 0$$

Rational expressions raised to a negative exponent can be simplified

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \quad a, b \neq 0$$

Evaluate Expressions Using Negative Exponents

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$6^{-2} \cdot 6^3 = 6^{-2+3} = 6^1 = 6$$

$$\begin{aligned} (-2)^{-4} + 4^{-2} &= \frac{1}{(-2)^4} + \frac{1}{4^2} \\ &= \frac{1}{16} + \frac{1}{16} = \frac{2}{16} = \frac{1}{8} \end{aligned}$$

$$(4^{-2})^{-3} \div 4^8 = 4^{(-2)(-3)} \div 4^8$$

$$= 4^6 \div 4^8$$

$$= 4^{6-8}$$

$$= 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$$

Simplify Expressions With Negative Exponents

$$(x^2)(x^{-3})(x^4) = x^{-2 + -3 + 4} = x^{-1} = \frac{1}{x}$$

$$\frac{a^2 b^{-3}}{a^{-1} b^2} = a^{2 - (-1)} b^{-3 - 2} = a^3 b^{-5} = \frac{a^3}{b^5}$$

$$(2u^3v^{-2})^{-3} = (2)^{-3} u^{(3)(-3)} v^{(-2)(-3)} \\ = \frac{1}{2^3} u^{-9} v^6 \\ = \frac{v^6}{8u^9}$$

Evaluate Powers Involving fractional Bases

$$(\frac{1}{3})^{-1} = \frac{1}{(\frac{1}{3})^1} = 1 \cdot \frac{3}{1} = 3$$

$$\left(\frac{-27}{8}\right)^{-2} = \left(\frac{-8}{27}\right)^2 = \frac{64}{729}$$

$$\therefore \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

Rational Exponents

3.3

Evaluate

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2 \quad \left(\frac{1}{n}\right)$$

$$(-32)^{\frac{1}{5}} = \sqrt[5]{-32} = -2$$

$$-16^{\frac{1}{4}} = -\sqrt[4]{16} = -2$$

$$(-27)^{-\frac{1}{3}} = \frac{1}{(-27)^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{-27}} = \frac{1}{-3}$$

Evaluate

$\left(\frac{m}{n}\right)$

$$8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2$$

$$= (2)^2$$

$$= 4$$

$$\left(\frac{49}{81}\right)^{-\frac{3}{2}} = \left(\frac{81}{49}\right)^{\frac{3}{2}}$$

$$= \left(\sqrt{\frac{81}{49}}\right)^3$$

$$= \left(\frac{9}{7}\right)^3$$

$$= \frac{729}{343}$$

$$81^{\frac{5}{4}} = \left(\sqrt[4]{81}\right)^5$$

$$= 3^5$$

$$= 243$$

Simplify

$$\frac{(x^{\frac{2}{3}})(x^{\frac{2}{3}})}{x^{\frac{1}{3}}} = \frac{x^{\frac{2}{3} + \frac{2}{3}}}{x^{\frac{1}{3}}} = \frac{x^{\frac{4}{3}}}{x^{\frac{1}{3}}} = x^{\frac{4}{3} - \frac{1}{3}} = x^{\frac{3}{3}} = x$$

$$(y^{\frac{1}{4}})^2 \cdot (y^{-\frac{1}{3}})^2 = y^{\frac{2}{4}} \cdot y^{-\frac{2}{3}}$$

$$= y^{\frac{2}{4} + -\frac{2}{3}} \\ = \frac{y^{\frac{6}{12}}}{y^{\frac{8}{12}}} = \frac{y^{\frac{2}{4}}}{y^{\frac{8}{12}}}$$

$$= y^{\frac{2}{4}}$$

$$= y^{-\frac{2}{12}}$$

$$= y^{-\frac{1}{4}}$$

$$= \frac{1}{y^{\frac{1}{4}}}$$

Properties of Exponential functions

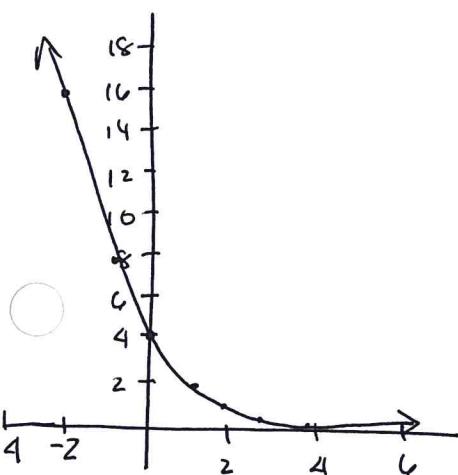
3.4

- Graph the function.

Identify the domain, range, $x + y$ intercepts, intervals of increase / decrease, asymptote

$$y = 4 \left(\frac{1}{2}\right)^x$$

x	y
-2	16
-1	8
0	4
1	2
2	1
3	$\frac{1}{2}$
4	$\frac{1}{4}$



domain: $\{x \in \mathbb{R}\}$ function is defined for all values of x

range: $\{y \in \mathbb{R}, y > 0\}$ function has positive values for y
but never reaches zero

no x -intercept

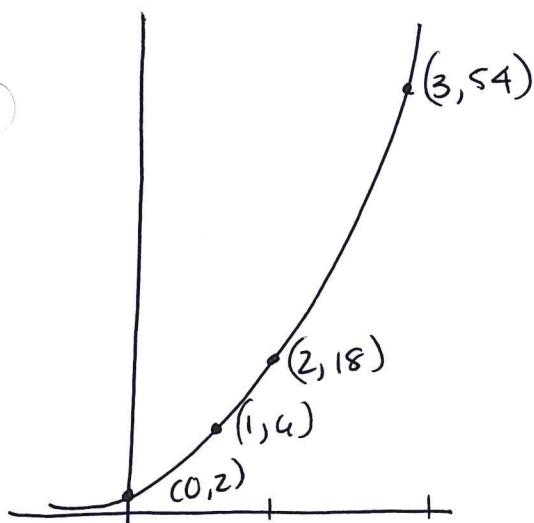
y intercept = 4

graph falls to the right throughout domain ∵

function is decreasing over domain

As x values increase y values decrease but never
reach 0. ∴ x axis is asymptote $y = 0$

Write an Exponential Equation given its graph



x	y	$\frac{\Delta y}{x}$
0	2	
1	6	
2	18	
3	54	

Write an equation in the form $y = ab^x$ for the graph

since $\Delta y = 3$ $b = 3$

to find "a" substitute in a point

$$y = ab^x \quad \text{use } (1, 6)$$

$$y = a \cdot 3^x \quad 6 = a \cdot 3^1$$

$$6 = 3a$$

$$2 = a$$

$$\therefore y = 2 \cdot 3^x$$

Write an exponential function given its properties

A radioactive sample has a half-life of 3 days. The initial sample is 200mg.

Write a function to relate the amount remaining, in mg, to the time, t days.

$$A(x) = A_0 \left(\frac{1}{2}\right)^x$$

$A(x)$ = amount after x days

A_0 = initial amount

x = days in $\frac{1}{2}$ life periods

$$A(x) = 200 \left(\frac{1}{2}\right)^x$$

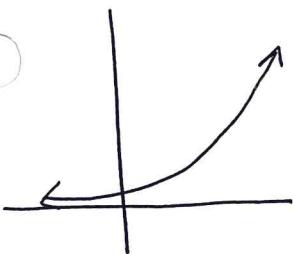
$\frac{1}{2}$ life = 3 days so substitute $\frac{t}{3}$ for x

because the number of elapsed $\frac{1}{2}$ lives is the number of days divided by 3

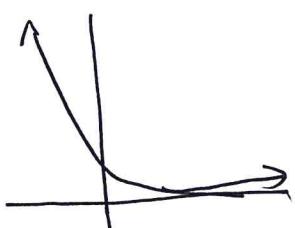
This relates it to t = time in days

$$A(t) = 200 \left(\frac{1}{2}\right)^{t/3}$$

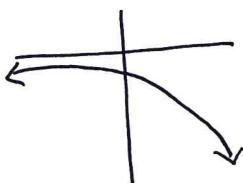
Graphs of exponential functions



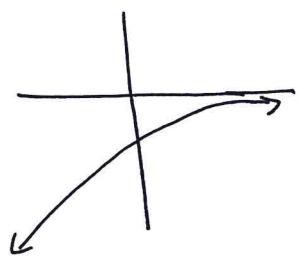
increasing
 $a > 0$
 $b > 1$



decreasing
 $a > 0$
 $0 < b < 1$



decreasing
 $a < 0$
 $b > 1$



increasing
 $a < 0$
 $0 < b < 1$

in the graph $y = ab^x$ where $a > 0 \rightarrow b > 0$

domain $\{x \in \mathbb{R}\}$

horizontal asymptote at $y = 0$

range $\{y \in \mathbb{R}, y > 0\}$

y-intercept = a

in the graph $y = ab^x$ where $a < 0$ and $b > 0$

domain $\{x \in \mathbb{R}\}$

horizontal asymptote at $y = 0$

range $\{y \in \mathbb{R}, y < 0\}$

y-intercept = a

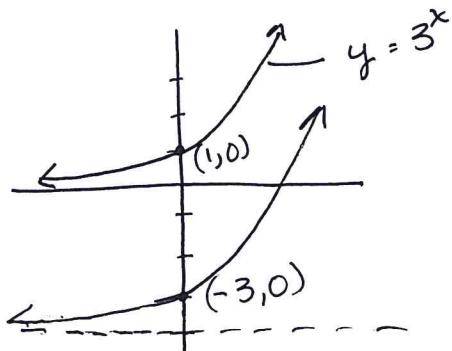
Transformations of exponential functions

3.5

Vertical and horizontal translations

- Sketch the graphs using $y = 3^x$ as base
describe effects on domain, range, asymptote

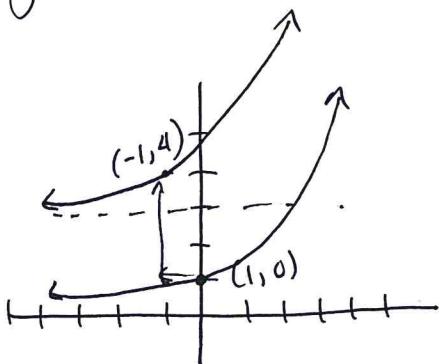
$$y = 3^x - 4 \quad \text{this will shift down 4 units}$$



asymptote shifted down 4 units
domain $\{x \in \mathbb{R}\}$ stays the same
range $\{y \in \mathbb{R}; y > -4\}$ instead of $y > 0$

y will change

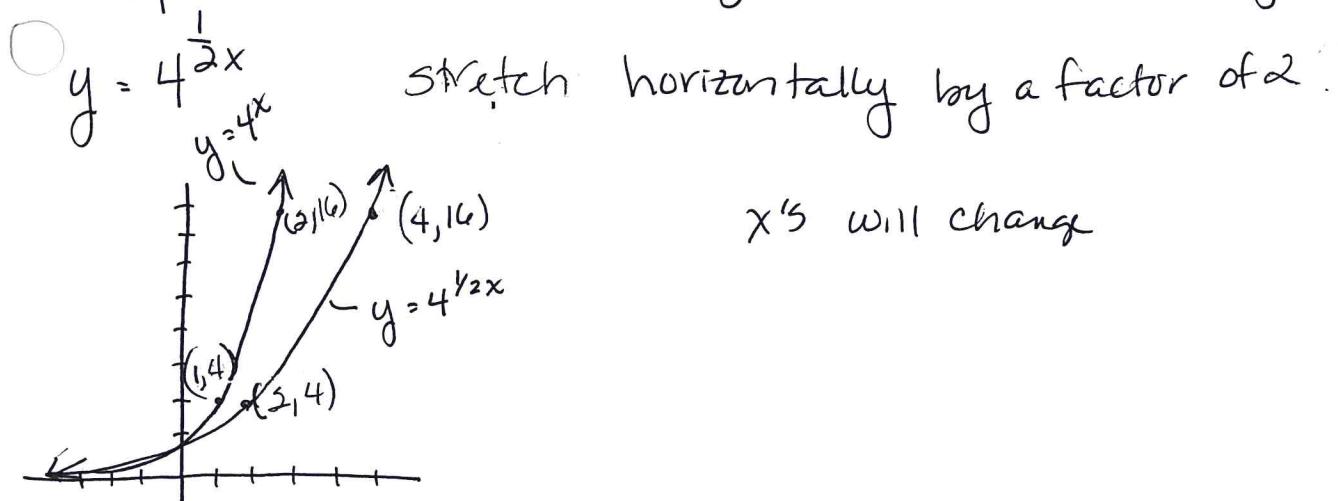
$$y = 3^{x+1} + 3 \quad \text{left one unit and up 3 units}$$



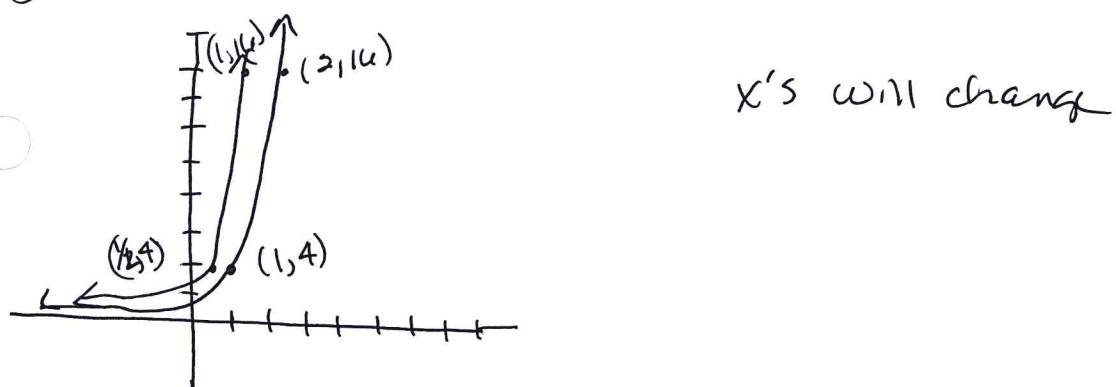
asymptote $y = 3$
domain $\{x \in \mathbb{R}\}$ same
range $\{y \in \mathbb{R}, y > 3\}$

Stretches and Compressions

Graph the function using the base function $y = 4^x$



$y = 4^{2x}$ compress horizontally by a factor of $\frac{1}{2}$

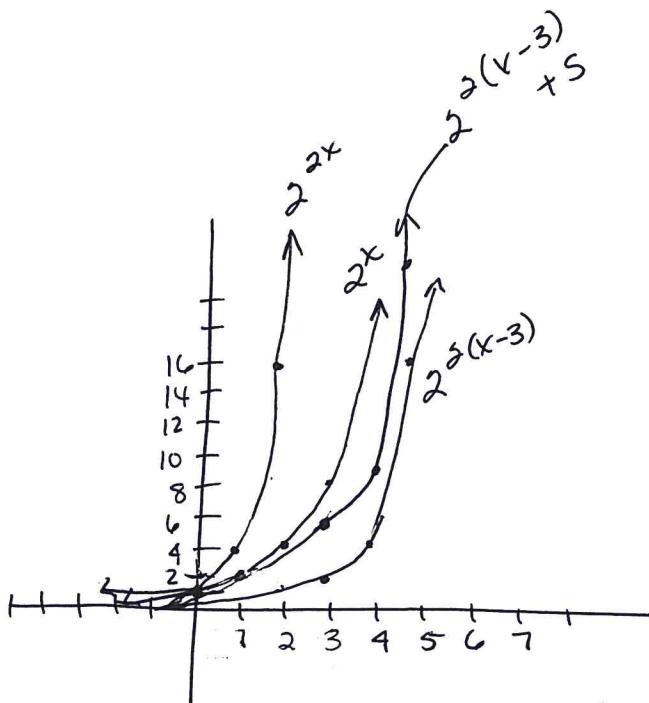


Graph $y = ab^{k(x-d)} + c$

$$y = 2^{2(x-3)} + 5$$

Start with 2^x

- Compress horizontally by factor of $\frac{1}{2}$
- Shift right 3
- Shift up 5



2^x	x	y
	0	1
	1	2
	2	4
	3	8

2^{2x}	x	y
	0	1
	1	4
	2	16
	3	64

$2^{2(x-3)}$	x	y
	3	1
	4	4
	5	16
	6	64

$2^{2(x-3)} + 5$	x	y
	3	6
	4	9
	5	21
	6	69

Translations

$$y = ab^{k(x-d)} + c$$

Compare to $y = b^x$

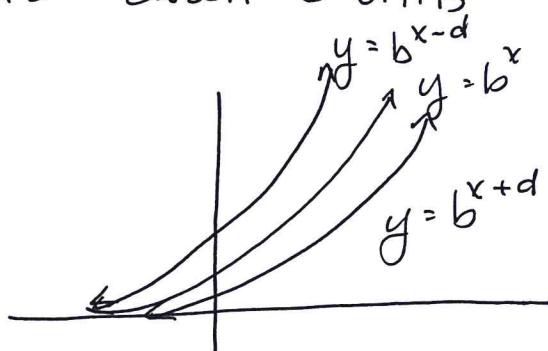
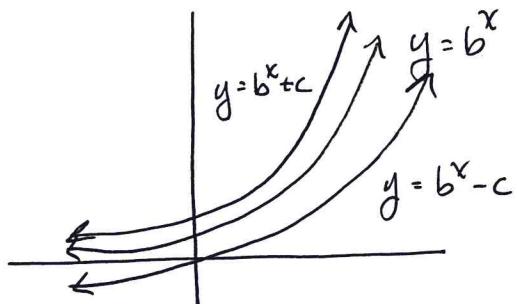
horizontal & vertical translations

$d > 0$ right d units

$c > 0$ up c units

$d < 0$ left d units

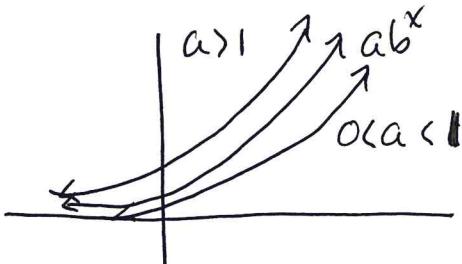
$c < 0$ down c units



Vertical stretches and compressions

$a > 1$ stretch vertically by a factor of " a "

$0 < a < 1$ compress vertically by a factor of " a "



horizontal stretch and compression

$k > 1$ compress horizontally by a factor of $\frac{1}{k}$

$0 < k < 1$ stretch horizontally by a factor of k

